WEEK 2

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February 13, 2015
Objectives: locate the dominant eigenvalue, power method, inverse power method

1 The Power Method

Let $A$ be $n \times n$, $\{\lambda\}^n_{i=1}$ its eigenvalues and $\{v^{(i)}\}^n_{i=1}$ its associated LI eigenvectors. Let the eigenvalues be ordered

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq ... \geq |\lambda_n|$$

where $\lambda_1$ is called the dominant eigenvalue of $A$ (largest magnitude).

Any vector $x$ can be written as

$$x = \sum_{j=1}^{n} \beta_j v^{(j)}$$

where the $\beta_i$’s are real numbers. If $A$ is applied on both sides one gets

$$Ax = \sum_{j=1}^{n} \beta_j A v^{(j)} = \sum_{j=1}^{n} \beta_j \lambda_j v^{(j)}$$

Applying $A$ 2-times one gets

$$A^2x = \sum_{j=1}^{n} \beta_j A^2 v^{(j)} = \sum_{j=1}^{n} \beta_j \lambda_j^2 v^{(j)}$$

Applying $A$ $k$-times one gets

$$A^kx = \sum_{j=1}^{n} \beta_j A^k v^{(j)} = \sum_{j=1}^{n} \beta_j \lambda_j^k v^{(j)}$$

Factoring $\lambda_1^k$ out, one gets

$$A^kx = \left( \lambda_1^k \beta_1 v^{(1)} + \sum_{j=2}^{n} \beta_j \left( \frac{\lambda_j}{\lambda_1} \right)^k v^{(j)} \right)$$

Since $\lambda_1$ is the largest eigenvalue

$$\lim_{k \to \infty} A^kx = \lim_{k \to \infty} \lambda_1^k \beta_1 v^{(1)}$$

At this stage, scaling is needed to get an algorithm for approximating $\lambda_1$.

Example 1. consider the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$$
with eigenvalues $\lambda_1 = 4$ and $\lambda_2 = -1$. We start with $x^0 = (1, 0)^t$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$x^{k+1} = A^k x^k$</th>
<th>$\mu_{k+1} = \frac{\max x^{k+1}_i}{\max x^k_i}$</th>
<th>$\mu_{k+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$(1, 0)^t$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$(1, 2)^t$</td>
<td>2/1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>$(7, 6)^t$</td>
<td>7/2</td>
<td>3.5</td>
</tr>
<tr>
<td>3</td>
<td>$(25, 26)^t$</td>
<td>26/7</td>
<td>3.7143</td>
</tr>
<tr>
<td>4</td>
<td>$(103, 102)^t$</td>
<td>103/26</td>
<td>3.9615</td>
</tr>
<tr>
<td>5</td>
<td>$(409, 410)^t$</td>
<td>410/103</td>
<td>3.9806</td>
</tr>
<tr>
<td>6</td>
<td>$(1639, 1638)^t$</td>
<td>1639/410</td>
<td>3.9976</td>
</tr>
<tr>
<td>7</td>
<td>$(6553, 6554)^t$</td>
<td>6554/1963</td>
<td>3.9988</td>
</tr>
<tr>
<td>8</td>
<td>$(26215, 26214)^t$</td>
<td>26215/6554</td>
<td>3.9998</td>
</tr>
</tbody>
</table>

Remark that $\lim_{k \to \infty} \mu_k = 4$.

Select a unit vector $x^{(0)}$ with unit $l_\infty$ norm and a unit component $x^{(0)}_{p_0}$ of $x^{(0)}$:

$$x^{(0)} = \left( x^{(0)}_1, \ldots, x^{(0)}_{p_0}, \ldots, x^{(0)}_n \right)^t \quad \text{and} \quad \|x^{(0)}\|_\infty = \|x^{(0)}_{p_0}\| = 1.$$

Now let

$$\begin{align*}
y^{(1)} &= A x^{(0)} \\
\mu^{(1)} &= y^{(1)}_{p_0} = \frac{x^{(0)}_{p_0}}{x^{(0)}_{p_0}} = \lambda_1 \beta_1 v^{(1)}_{p_0} + \sum_{j=2}^{n} \beta_j \left( \frac{\lambda_j}{\lambda_1} \right) v^{(j)}_{p_0}.
\end{align*}$$

Then let $p_1$ be the least integer such that $|y^{(1)}_{p_1}| = \|y^{(1)}\|_\infty$ and define

$$x^{(1)} = \frac{1}{y^{(1)}_{p_1}} y^{(1)} = \frac{1}{y^{(1)}_{p_1}} A x^{(0)}.$$

Define now in the next iteration $y^{(2)}$ as

$$\begin{align*}
y^{(2)} &= A x^{(1)} = \frac{1}{y^{(1)}_{p_1}} A^2 x^{(0)} \\
\mu^{(2)} &= y^{(2)}_{p_1} = \frac{x^{(1)}_{p_1}}{x^{(1)}_{p_1}} = \lambda_1 \beta_1 v^{(1)}_{p_1} + \sum_{j=2}^{n} \beta_j \left( \frac{\lambda_j}{\lambda_1} \right)^2 v^{(j)}_{p_1}.
\end{align*}$$

Then let $p_2$ be the least integer such that $|y^{(2)}_{p_2}| = \|y^{(2)}\|_\infty$ and define

$$x^{(2)} = \frac{1}{y^{(2)}_{p_2}} y^{(2)} = \frac{1}{y^{(2)}_{p_2}} A x^{(1)}.$$
Thus, sequences of vectors \( \{\mathbf{x}^{(m)}\}_{m=0}^{\infty} \) and \( \{\mathbf{y}^{(m)}\}_{m=1}^{\infty} \) and of scalars \( \{\mu^{(m)}\}_{m=1}^{\infty} \) can be produced by

\[
\begin{align*}
\mathbf{y}^{(m)} &= A\mathbf{x}^{(m-1)} \\
\mu^{(m)} &= \frac{y_{p_m-1}^{(1)}}{\sum_{j=2}^{n} \beta_j \lambda_j / \lambda_1} + \sum_{j=2}^{n} \beta_j \left( \frac{\lambda_j / \lambda_1}{\lambda_1} \right) v^{(j)}_{p_m-1} \\
\mathbf{x}^{(m)} &= \frac{1}{y_{p_m}^{(m)}} \mathbf{y}^{(m)},
\end{align*}
\]

where at each step \( p_m \) is the smallest integer for which

\[
\begin{align*}
\mathbf{y}^{(m)} &= (y_{m}^{(m)}, \ldots, y_{p_m}^{(m)}, \ldots, y_{n}^{(m)})^t \\
|y_{p_m}^{(m)}| &= \|\mathbf{y}^{(m)}\|_{\infty}.
\end{align*}
\]

Because of the dominance of the first eigenvalue

\[
\begin{align*}
\lim_{m \to \infty} \mu^{(m)} &= \lambda_1 \\
\lim_{m \to \infty} \mathbf{x}^{(m)} &= \mathbf{v}^1 \text{ eigenvector associated with } \lambda_1.
\end{align*}
\]

**Algorithm 1.**

Given \( A, \mathbf{x} \) and a stopping criterion \( \epsilon \), let \( x^{(0)} = \frac{1}{\|\mathbf{x}\|_{\infty}} \). Because \( \|x^{(0)}\|_{\infty} = 1 \), let \( |x_{p_0}| = \|x^{(0)}\|_{\infty} = 1 \) Approximate \((\lambda_1, \mathbf{v}_1)\) as follows: For \( k = 1, 2, \ldots \)

1. Compute \( y^{(k)} = A x^{(k-1)} \).
2. Compute \( p_k \) such that \( |y_{p_k}^{(k)}| = \|y^{(k)}\|_{\infty} \).
3. Let \( r_k = y_{p_k}^{(k)} \) and \( x^{(k)} = \frac{1}{r_k} y^{(k)} \) (i.e \( \|x^{(k)}\|_{\infty} = 1 \)).
4. If \( \|x^{(k)} - x^{(k-1)}\|_{\infty} \leq \epsilon \), then \( \lambda_1 \simeq r_k \) and \( \mathbf{v}_1 = x^{(k)} \), then stop. Otherwise, repeat the steps 1-4 for \( k + 1 \).

**Example 2.** Consider the matrix

\[
A = \begin{pmatrix}
-2 & -2 & 3 \\
-10 & -1 & 6 \\
10 & -2 & -9
\end{pmatrix}
\]

whose eigenvalues are \( \lambda_1 = -12, \lambda_2 = -3, \) and \( \lambda_3 = 3 \). Let start with \( \mathbf{x}^{(0)} \) such that

\[
\mathbf{x}^{(0)} = (1, 0, 0)^t; \quad p_0^{(0)} = 1
\]
First iteration

\[ y^{(1)} = Ax^{(0)} = (-2, -10, 10)^t; \]
\[ p_1 = 2; \quad x_{p_1}^{(1)} = 2; \quad y_1^{(1)} = \lambda_1^{(1)} = -2 \]
\[ x^{(1)} = \frac{1}{-10} y^{(1)} = (-0.2, -1, 1)^t \]

Second iteration

\[ y^{(2)} = Ax^{(1)} = (-27/5, -9, 9)^t; \]
\[ p_2 = 2; \quad x_{p_2}^{(2)} = 2; \quad y_2^{(2)} = \lambda_2^{(2)} = -9. \]
\[ x^{(2)} = \frac{1}{-9} y^{(2)} = (-3/5, 1, -1)^t. \]

Third iteration

\[ y^{(3)} = Ax^{(2)} = (-31/5, -13, 13)^t; \]
\[ p_3 = 2; \quad x_{p_3}^{(3)} = 2; \quad y_3^{(3)} = \lambda_3^{(3)} = -13 \]
\[ x^{(2)} = \frac{1}{-13} y^{(2)} = (0.4769; -1; 1)^t. \]

<table>
<thead>
<tr>
<th>iter.</th>
<th>eigenvector</th>
<th>eigenvalue</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-\frac{1}{10}.0000) ((-2.0000; -10.0000; 10.0000)^t)</td>
<td>(\lambda^{(1)} = -2)</td>
</tr>
<tr>
<td>2</td>
<td>(-\frac{1}{9}.0000) ((-5.4000; -9.0000; 9.0000)^t)</td>
<td>(\lambda^{(2)} = -9)</td>
</tr>
<tr>
<td>3</td>
<td>(-\frac{1}{13}.0000) ((-6.2000; -13.0000; 13.0000)^t)</td>
<td>(\lambda^{(3)} = -13)</td>
</tr>
<tr>
<td>4</td>
<td>(-\frac{1}{11.7692}) ((-5.9538; -11.7692; 11.7692)^t)</td>
<td>(\lambda^{(4)} = -11.7692)</td>
</tr>
<tr>
<td>5</td>
<td>(-\frac{1}{12.9021}) ((-6.4322; -12.9021; 12.9021)^t)</td>
<td>(\lambda^{(5)} = -12.9021)</td>
</tr>
<tr>
<td>6</td>
<td>(-\frac{1}{11.9854}) ((-5.9971; -11.9854; 11.9854)^t)</td>
<td>(\lambda^{(6)} = -11.9854)</td>
</tr>
<tr>
<td>7</td>
<td>(-\frac{1}{12.0036}) ((-6.0007; -12.0036; 12.0036)^t)</td>
<td>(\lambda^{(7)} = -12.0036)</td>
</tr>
<tr>
<td>8</td>
<td>(-\frac{1}{11.9991}) ((-5.9998; -11.9991; 11.9991)^t)</td>
<td>(\lambda^{(8)} = -11.9991)</td>
</tr>
<tr>
<td>9</td>
<td>(-\frac{1}{12.0002}) ((-6.0001; -12.0002; 12.0002)^t)</td>
<td>(\lambda^{(9)} = -12.0002)</td>
</tr>
</tbody>
</table>
2 Convergence Theorems

**Theorem 1 (Power Method).**

Assume that the matrix $A \in \mathbb{R}^{n \times n}$ has $n$ distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and that they are ordered in decreasing magnitude; that is, $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$. If $x^{(0)}$ is chosen appropriately, then the sequences $x^{(0)} = (x_1^{(0)}, \ldots, x_n^{(0)})$ and $r_k$ generated recursively by

$$y^{(k)} = Ax^{(k)}; \quad \text{and} \quad x^{(k+1)} = \frac{1}{r_{k+1}}x^{(k)}$$

where

$$r_{k+1} = y_{p_k}^{(k)} \quad \text{and} \quad |x_{p_k}^{(k)}| = \|y^{(k)}\|_\infty.$$

The sequence $r_k$ will converge to the dominant eigenvalue $\lambda_1$ with eigenvector $v_1$. That is,

$$\lim_{k \to \infty} r_k = \lambda_1 \quad \text{and} \quad \lim_{k \to \infty} x^{(k)} = v_1$$

**Proof.** Construction of the proof is similar the the technics above.

**The speed of the convergence**

The speed of convergence of $x^{(k)}$ to $v_1$ is governed by the terms $(\lambda_j/\lambda_1)^k, j = 2, \ldots, n$. Consequently, the rate of convergence is linear. Similarly, the convergence of the sequence of constants $r_k$ to $\lambda_1$ is linear. The Aitken $\Delta^2$ method can be used for any linearly convergent sequence $p_k$ to form a new sequence,

$$\hat{r}_k = \frac{(r_{k+1} - r_k)^2}{r_{k+2} - 2r_{k+1} + r_k}$$

that converges faster.
function [evector, evalue]=Power_Method(A,x,maxN,tol)
% A is the n by n input square matrix
% x: the starting eigenvector (n by 1)
% max_N is the max number of iteration
% tol: the error tolerance
[n,n] = size(A); rk = rand(1,n);
error = 1; k = 0;
fprintf('k lambda(k)
');
while (k < maxN & error > tol)
    y = A*x;
evalue = (rk*y)/(rk*x);
x = y/evalue;
    error = norm(A*x-evalue*x);
fprintf('%2d %4g 
',k+1 , evalue);
k = k+1;
end
DomnantEigenvalue=evalue
Eigenvector = x

Example 3. Find the dominant eigenvalue of $A$ by using the power method and the following initial guesses

$I$: $x_0 = (1,0,0,0)$  
$II$: $x_0 = (1,1,0,0)$  
$III$: $x_0 = (1,1,1,0)$

<table>
<thead>
<tr>
<th>k</th>
<th>I : $\lambda(k)$</th>
<th>II : $\lambda(k)$</th>
<th>III : $\lambda(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.914258</td>
<td>0.152675</td>
<td>-0.48947</td>
</tr>
<tr>
<td>2</td>
<td>-10.5784</td>
<td>131.832</td>
<td>-114.698</td>
</tr>
<tr>
<td>3</td>
<td>14.8261</td>
<td>-0.591455</td>
<td>-4.84304</td>
</tr>
<tr>
<td>4</td>
<td>-5.82974</td>
<td>-65.2272</td>
<td>-14.2993</td>
</tr>
<tr>
<td>5</td>
<td>-15.0051</td>
<td>-4.48977</td>
<td>-7.91334</td>
</tr>
<tr>
<td>6</td>
<td>-8.23006</td>
<td>-14.5343</td>
<td>-10.5773</td>
</tr>
<tr>
<td>7</td>
<td>-10.5205</td>
<td>-7.75444</td>
<td>-9.08374</td>
</tr>
<tr>
<td>8</td>
<td>-9.15679</td>
<td>-10.6578</td>
<td>-9.81599</td>
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<td>10</td>
<td><strong>-9.44728</strong></td>
<td><strong>-9.83804</strong></td>
<td><strong>-9.62627</strong></td>
</tr>
</tbody>
</table>
3 Exercises

**Group project 1A**: (Min 2 Students-Max 4 Students)
Describe the shifted power method to compute the eigenvalues and eigenvectors of a square matrix $A \in \mathbb{R}^{n \times n}$.

1. Describe the method?
2. Describe the algorithm?
3. Compute two examples?

A **Handwritten** report should be turned by the end 10/5/1436.

**Group project 1B**: (Min 2 Students-Max 4 Students)
Describe symmetric power method to approximate the eigenvalues and eigenvectors of a square matrix $A \in \mathbb{R}^{n \times n}$.

1. Describe the method?
2. Describe the algorithm?
3. Compute two examples?

A **Handwritten** report should be turned by the end 10/5/1436.

**Problem 1**. Use the power method to approximate the dominant eigenvalue of the following matrices

$$A = \begin{pmatrix} -7 & 2 \\ 8 & -1 \end{pmatrix}; \quad B = \begin{pmatrix} 2 & 3 \\ -2 & 1 \end{pmatrix}$$

**Problem 2**. Use the power method to approximate the dominant eigenvalue of the following symmetric matrices

$$A = \begin{pmatrix} 5 & -2 \\ -2 & 8 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}$$

**Problem 3**. Use the power method to approximate the dominant eigenvalue of the following matrices

$$A = \begin{pmatrix} 3 & 4 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$
4 References


   John Wiley N.Y.,


   McGraw-Hill, N.Y.

   John Wiley N.Y.